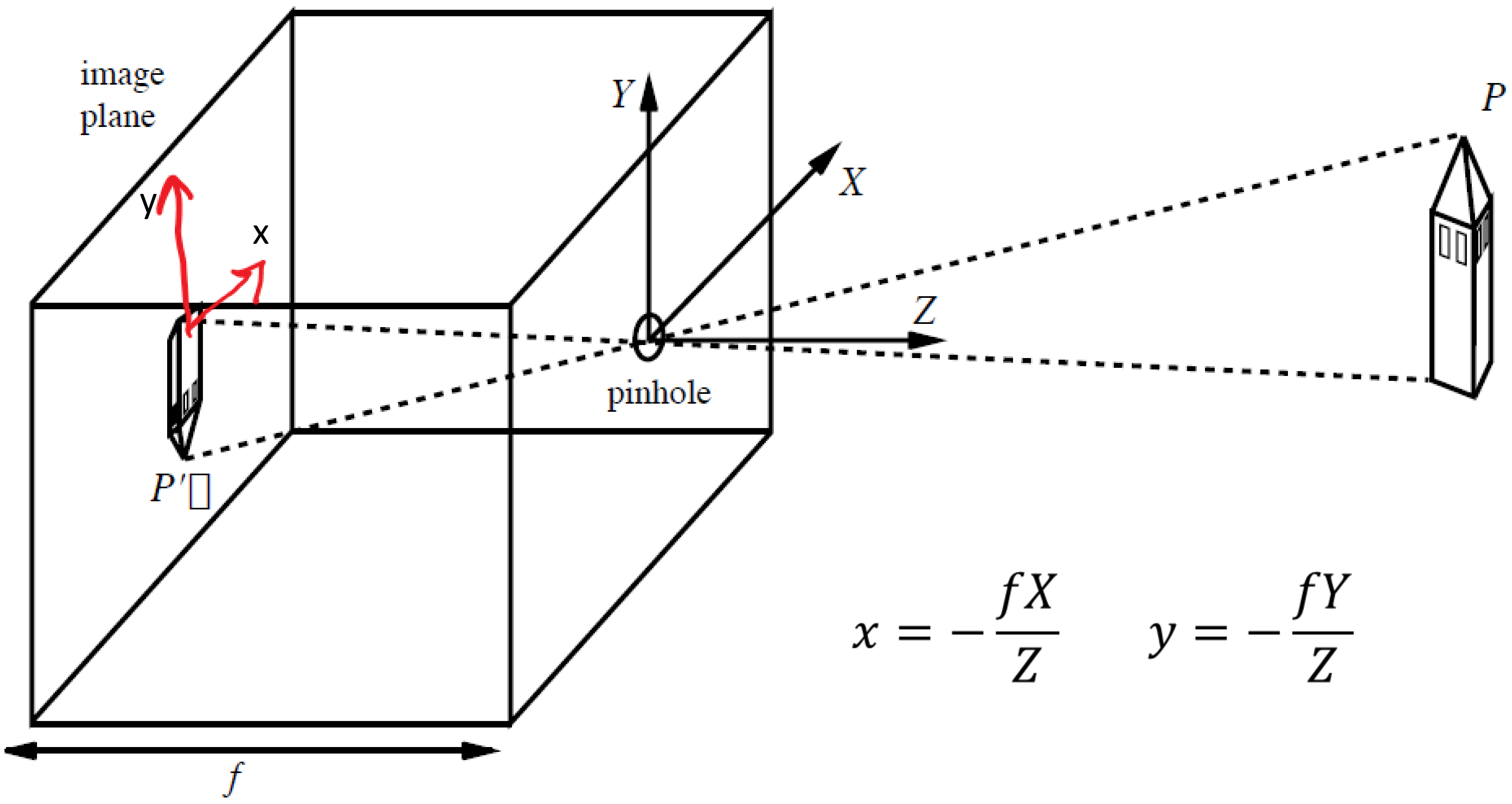
Camera Calibration

How we go from the pinhole model to real cameras imaging points in the 3D world

Important practical problem for which software can be found in OpenCV , Matlab, and many other places.

The Pinhole Camera



$$x = -\frac{fX}{Z} \qquad y = -\frac{fY}{Z}$$

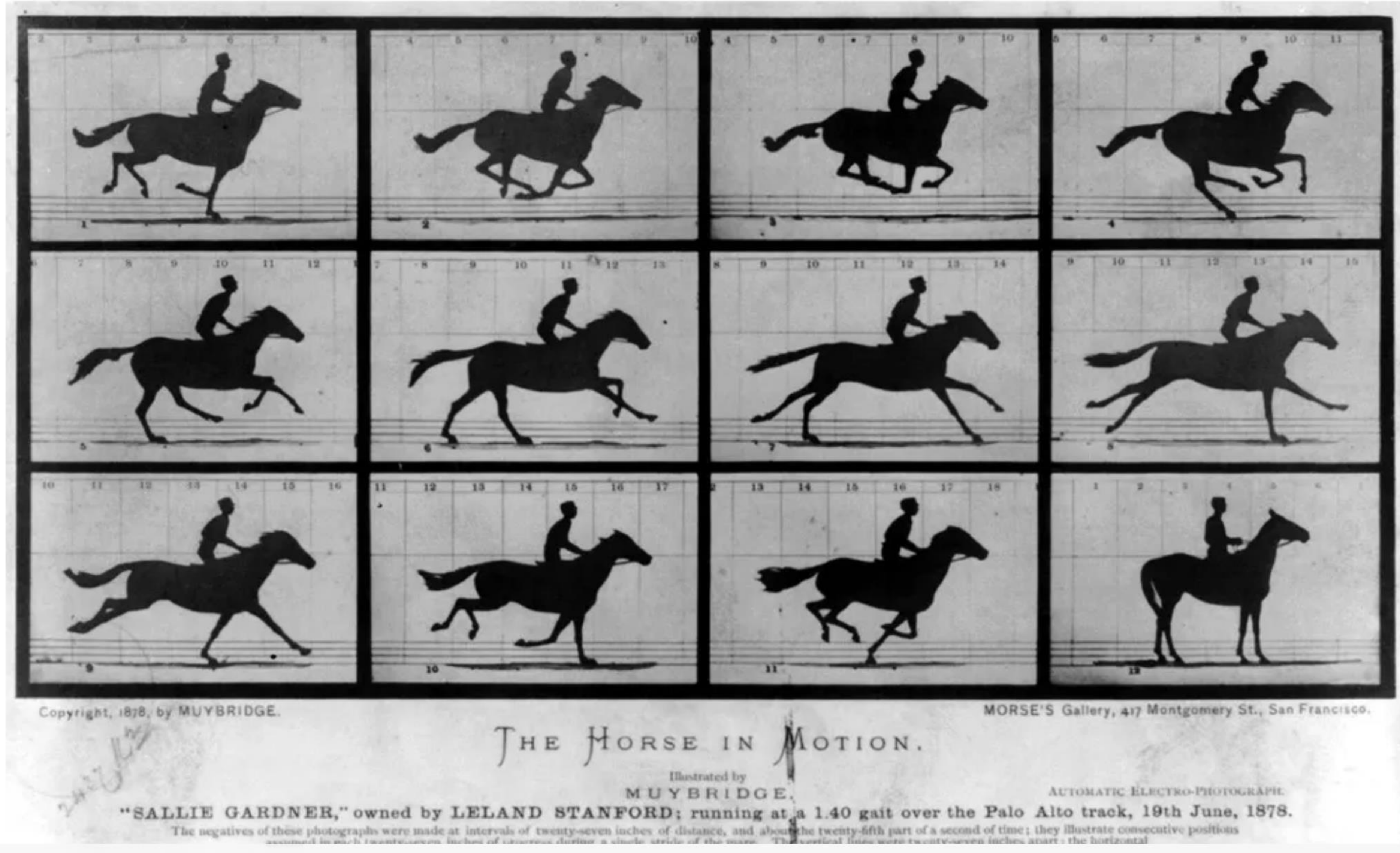
But how do we “preserve” this image?

Joseph Nicéphore Niépce

View from the Window at Le Gras, c. 1826

Louis Daguerre

Le Boulevard du Temple, Paris 1833

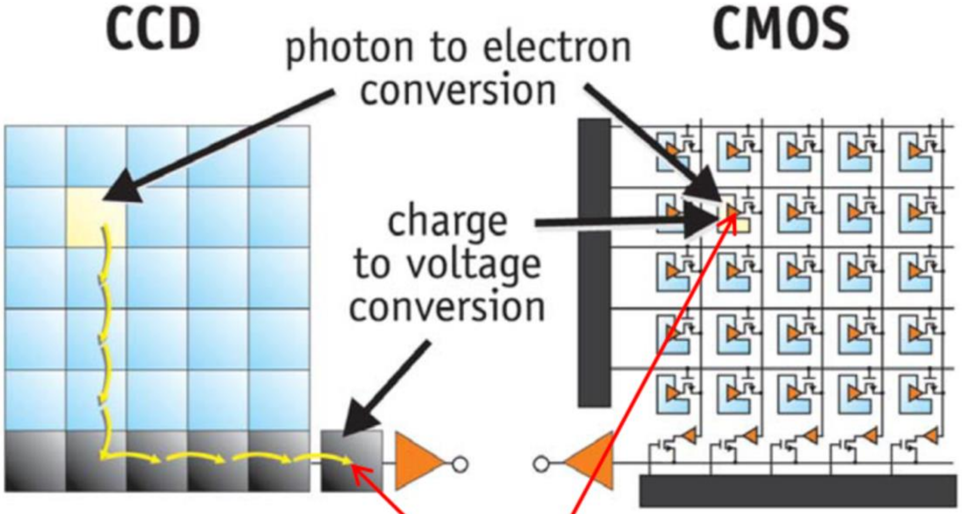


Eadweard Muybridge

The Horse in Motion, 1878

Digital Cameras

- The Charge Coupled Device was invented in 1969, based on exploiting the fact that silicon atoms can release electrons when hit by photons. Nowadays CMOS imagers are more common. The key difference is in how the charge generated is converted to a voltage (when digitized this becomes the pixel brightness value)



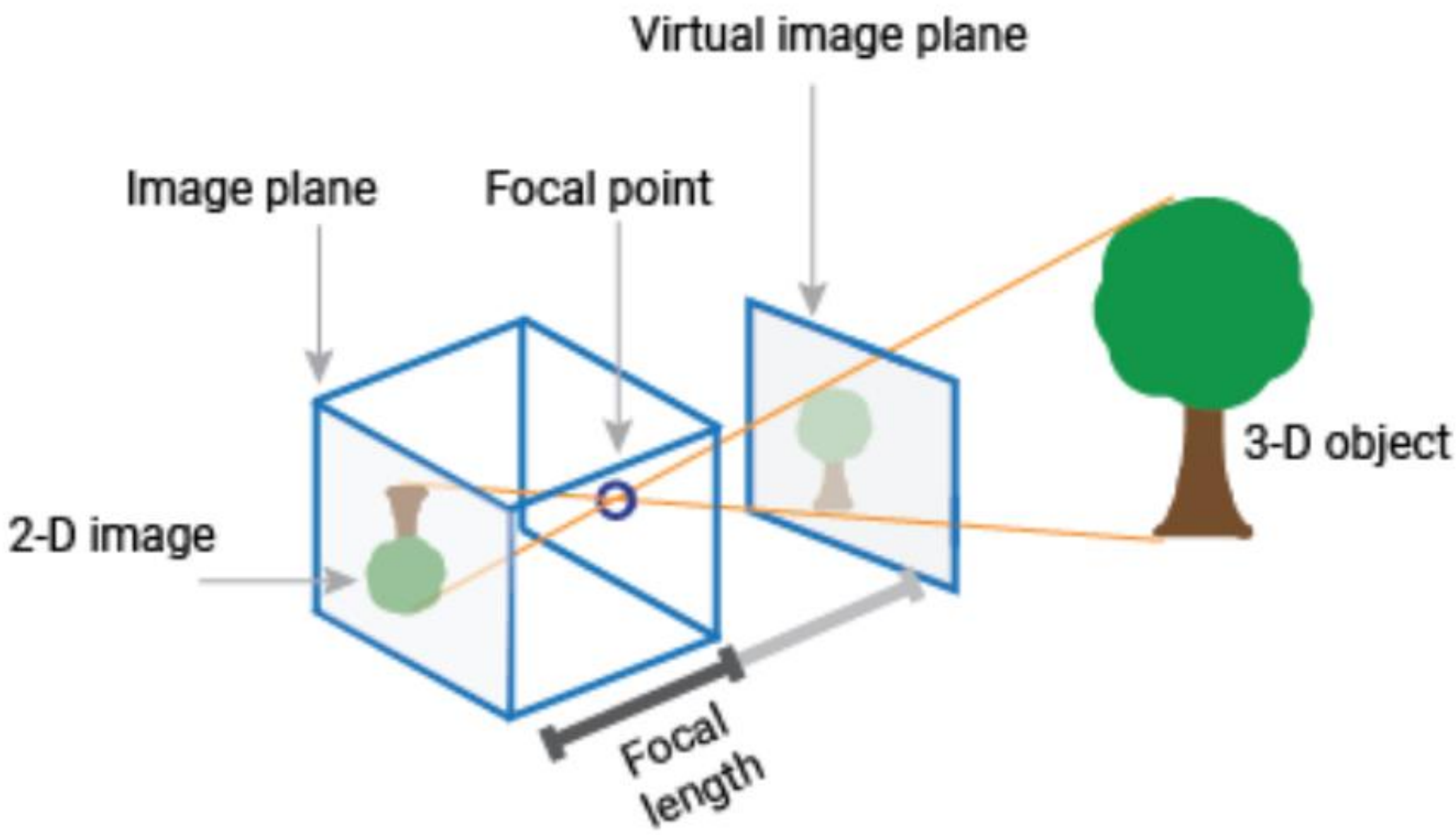
CCDs move photogenerated charge from pixel to pixel and convert it to voltage at an output node. CMOS imagers convert charge to voltage inside each pixel.

2009		Willard S. Boyle (1924–2011)	Canadian	"for the invention of an imaging semiconductor circuit – the CCD sensor"
		George E. Smith (b. 1930)	American	

Nobel Prize in Physics

Camera Calibration

(figures from https://www.mathworks.com/help/vision/ug/camera-calibration.html)



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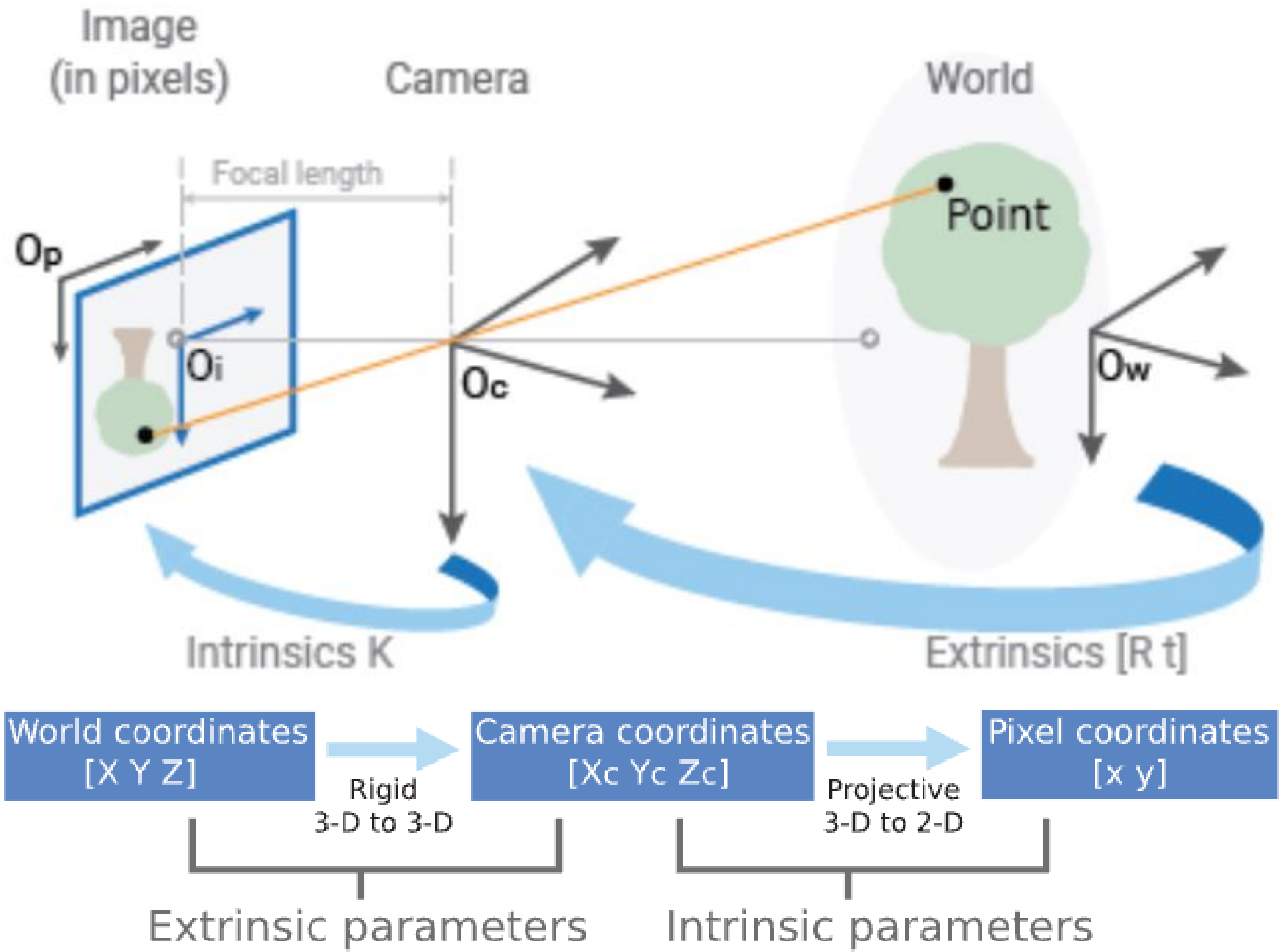

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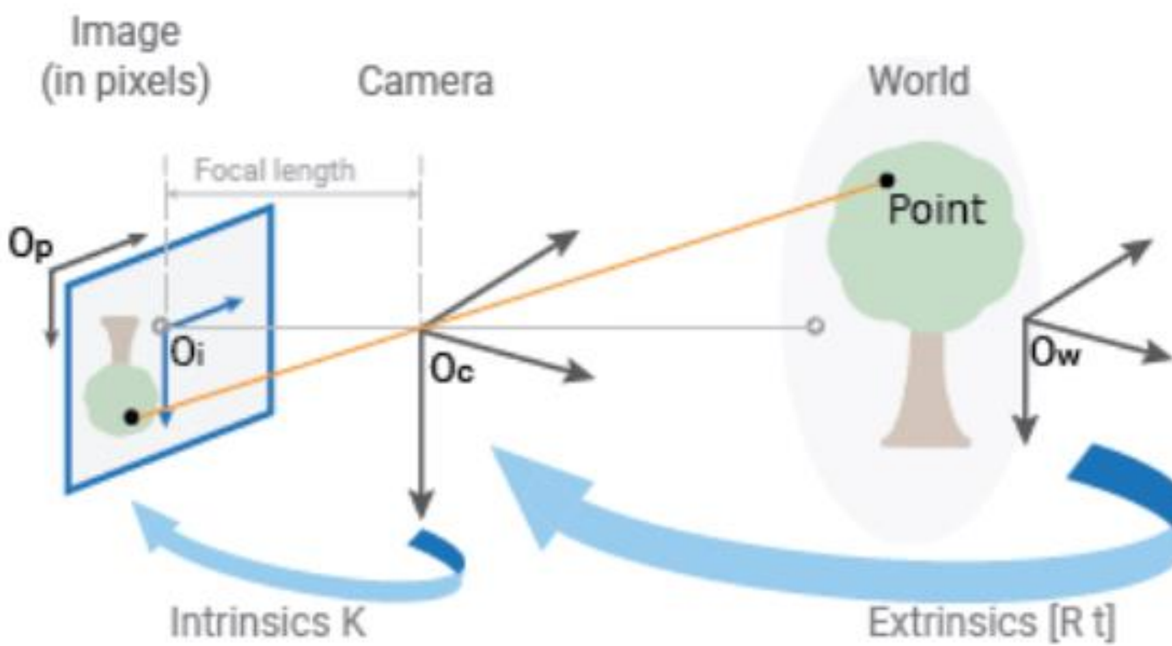


$$w\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = P\begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Scale factor

Image points

World points

$$P = K\begin{bmatrix} R & t \end{bmatrix}$$

Camera matrix

Intrinsics matrix

Extrinsics
Rotation and Translation

Intrinsic Parameters

(figures from https://www.mathworks.com/help/vision/ug/camera-calibration.html)

$$\begin{bmatrix} f_x & s & c_x \\ 0 & f_y & c_y \\ 0 & 0 & 1 \end{bmatrix}$$

$[c_x \quad c_y]$ – Optical center (the principal point), in pixels.

$(f_x, \quad f_y)$ – Focal length in pixels.

$f_x = F/p_x$

$f_y = F/p_y$

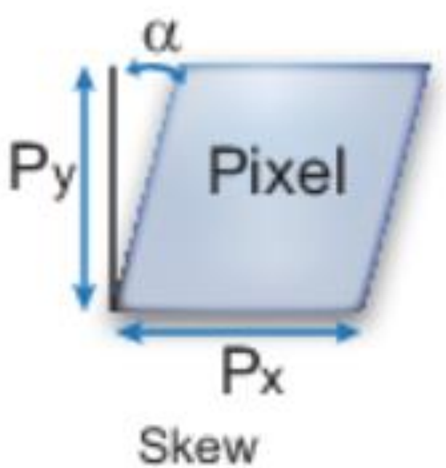
F – Focal length in world units, typically expressed in millimeters.

$(p_x, \quad p_y)$ – Size of the pixel in world units.

s – Skew coefficient, which is non-zero if the image axes are not perpendicular.

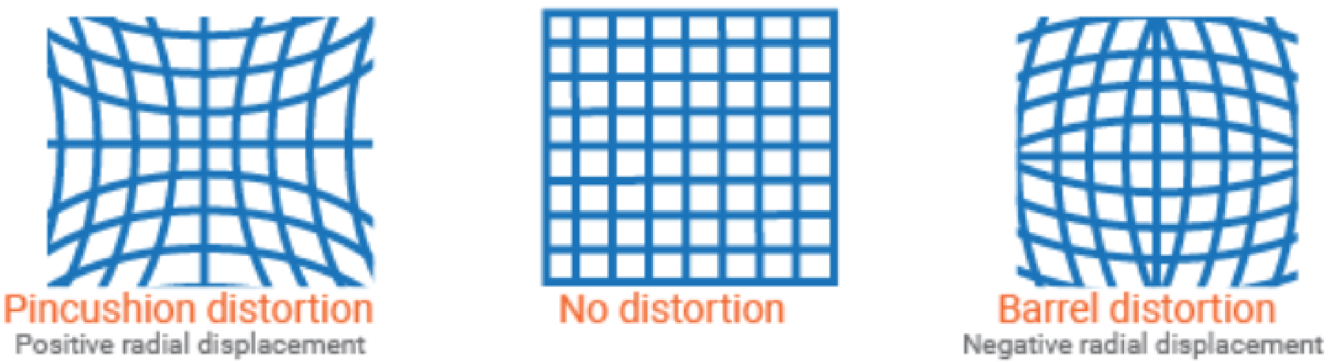
$s = f_x \tan \alpha$

The pixel skew is defined as:



Non-linear Lens Distortion

(figures from https://www.mathworks.com/help/vision/ug/camera-calibration.html)



The radial distortion coefficients model this type of distortion. The distorted points are denoted as ($x_{\text{distorted}}$, $y_{\text{distorted}}$):

$$x_{\text{distorted}} = x(1 + k_1{*}r^2 + k_2{*}r^4 + k_3{*}r^6)$$

$$y_{\text{distorted}} = y(1 + k_1{*}r^2 + k_2{*}r^4 + k_3{*}r^6)$$

Three types of transformations

- Rotation
- Translation
- Perspective projection

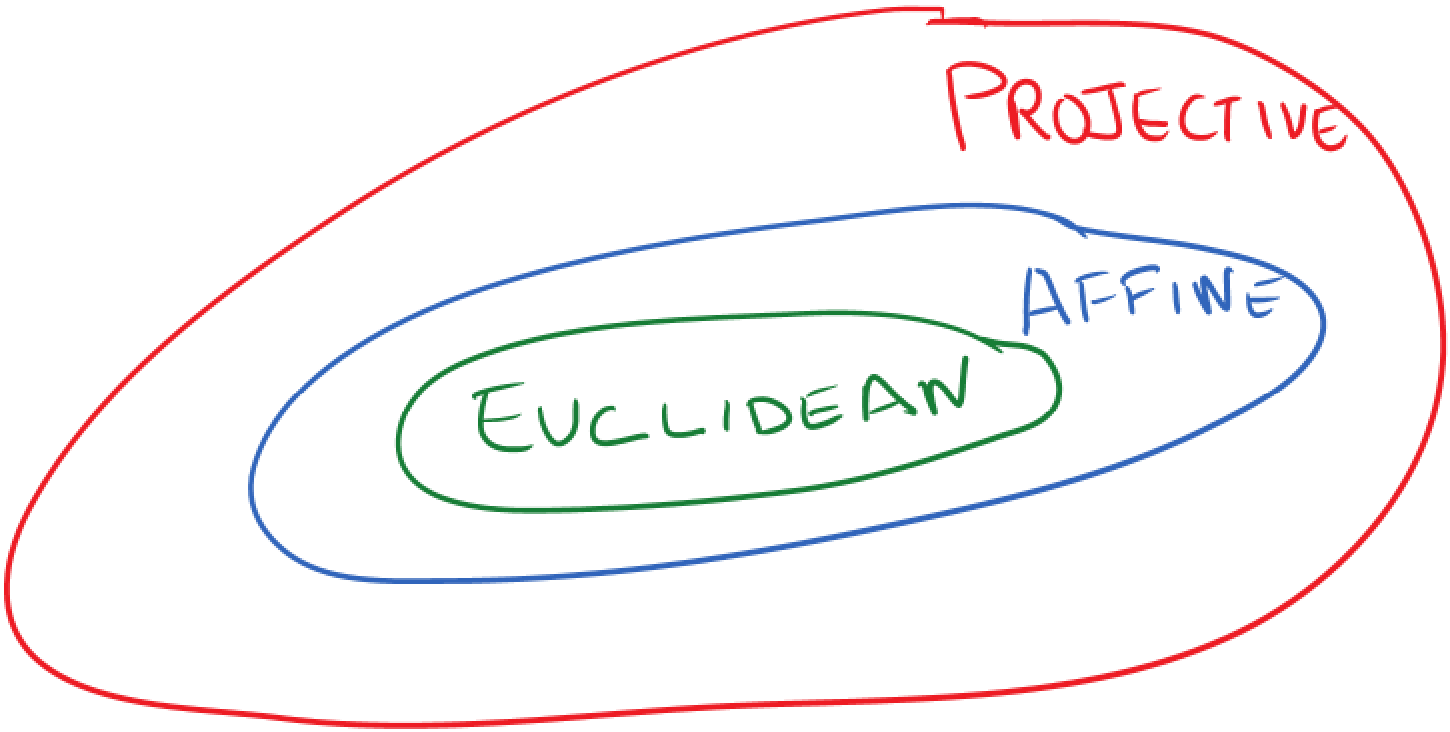
We like to use the tools of linear algebra, matrix-vector multiplication. But can we do that? Are these transformations linear?

Three types of transformations

- Rotation
- Translation
- Perspective projection

Homogeneous coordinates to the rescue!

Let us review projective transformations



Rigid body motions (Euclidean transformations / isometries)

- Theorem: Any rigid body motion can be expressed as an orthogonal transformation followed by a translation.

$$\psi(\mathbf{a}) = \mathbf{A}\mathbf{a} + \mathbf{t}$$

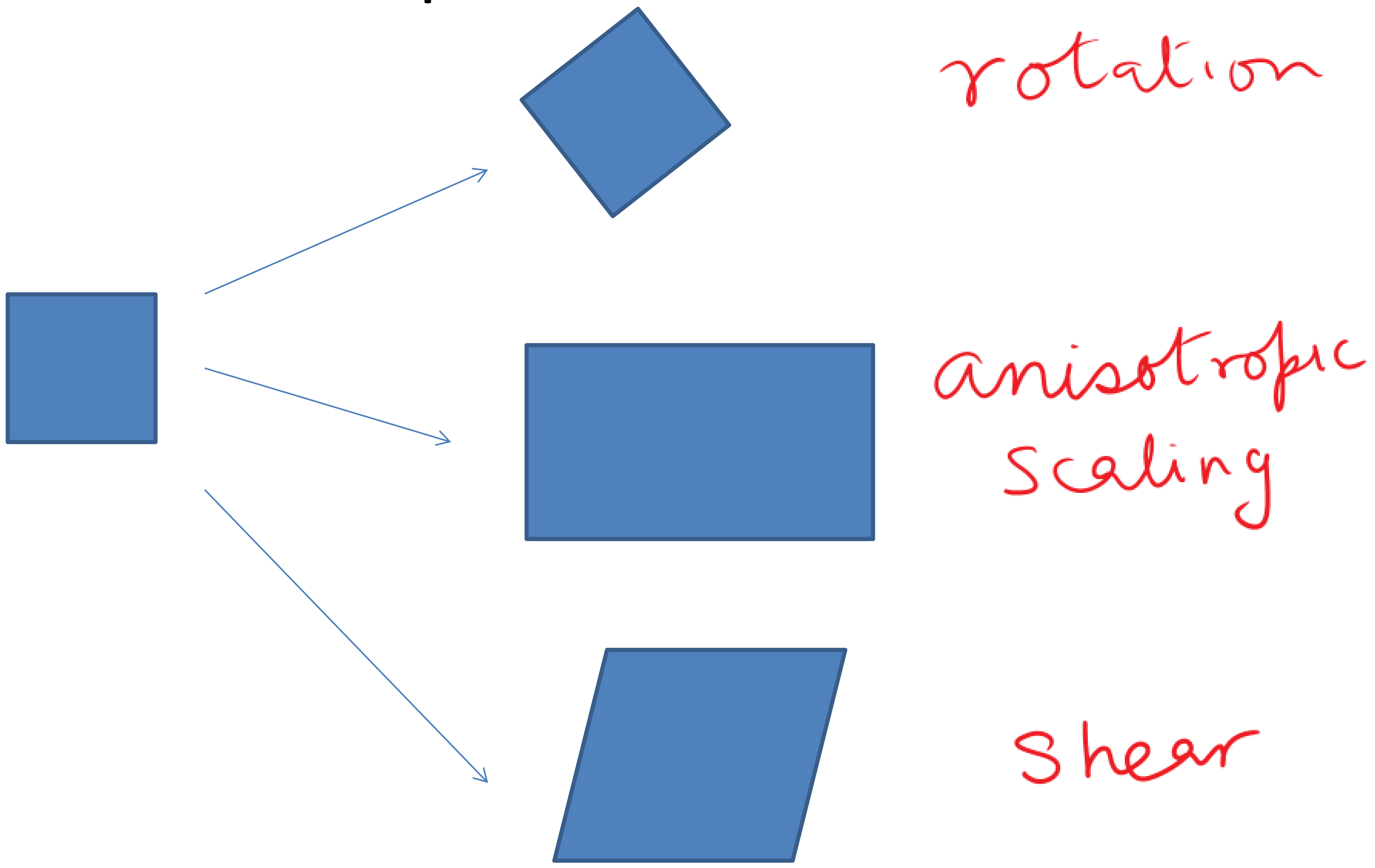
A is an orthogonal matrix

Affine transformations

- Definition: An affine transformation is a nonsingular linear transformation followed by a translation.

$$\psi(\mathbf{a}) = \mathbf{A}\mathbf{a} + \mathbf{t}$$

Some examples of affine transforms…



Projective Transformations

- Under perspective projection, parallel lines can map to lines that intersect. Therefore, this cannot be modeled by an affine transform!
- Projective transformations are a more general family which includes affine transforms and perspective projections.
- Projective transformations are linear transformations using homogeneous coordinates

Homogeneous coordinates

- Instead of using n coordinates for n-dimensional space, we use n+1 coordinates.

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ for P^1, the projective line

$\mathbb{R}^1 \cup \{\text{point at } \infty\}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ for P^2, the projective plane

$\mathbb{R}^2 \cup \{\text{line at } \infty\}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ for P^3 and so on...

Key rule

- $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $\begin{bmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \\ \lambda x_n \end{bmatrix}$ $\lambda \neq 0$

represent the same point in P^{n-1}
(thus only $n-1$ degrees of freedom)

Think of this as a line through the origin in n-dimensional space

Note: $x_1, x_2 \ldots x_n$ may not all be 0

Picking a canonical representative

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \longmapsto \begin{bmatrix} x_1/x_3 \\ x_2/x_3 \\ 1 \end{bmatrix}$$

(only if $x_3 \neq 0$)

$$\therefore (x, y) = \left(\frac{x_1}{x_3}, \frac{x_2}{x_3} \right)$$

Conversely

$$\begin{bmatrix} x \\ y \end{bmatrix} \longmapsto \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

augmented vector

The projective line

- Any finite point x can be represented as

$$\begin{bmatrix} x \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 2x \\ 2 \end{bmatrix} \text{ or } \begin{bmatrix} 6.3x \\ 6.3 \end{bmatrix} \text{ or} \ldots$$

- Any infinite point can be expressed as

$$\begin{bmatrix} x \\ 0 \end{bmatrix}$$

(Note: there is only one such point)

The projective plane

- Any finite point can be represented as

$$\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix}$$

- Any infinite point can be represented as

$$\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

Thus there is a line at infinity
Different ratios $x : y$ give different points.

Lines in homogeneous coordinates

Consider $a_1 x + a_2 y + a_3 = 0$

Note $\lambda a_1 x + \lambda a_2 y + \lambda a_3 = 0$ is the same line.

$$\begin{bmatrix} x \\ y \end{bmatrix} \longleftrightarrow \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$ with $x = \frac{X_1}{X_3}$, $y = \frac{X_2}{X_3}$

$$a_1 \frac{X_1}{X_3} + a_2 \frac{X_2}{X_3} + a_3 = 0$$

$$a_1 X_1 + a_2 X_2 + a_3 X_3 = 0$$

Incidence of points on lines

Q. When does a point $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ lie on a line $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$?

A.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Q. Where do the lines $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ intersect?

Incidence of points on lines

A. $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \wedge \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Example:

- $x = 1$ and $y = 1$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (1, 1)$$

Incidence of points on lines

A. $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \wedge \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Example:

- $x = 1$ and $x = 2$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \wedge \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Line incident on two points

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \text{ is given by}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \wedge \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \; !$$

An example of DUALITY

Representing affine transformations

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$x' = a_{11} x + a_{12} y + t_x$$

$$y' = a_{21} x + a_{22} y + t_y$$

$$w' = 1$$

which does the right thing!!

Euclidean transforms are affine, so the same trick works when $A = R$

Perspective Projection

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/f & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z/f \end{bmatrix}$$

$$= \begin{bmatrix} fX/Z \\ fY/Z \\ 1 \end{bmatrix}$$

Projective transformations

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$ in P^2

Notes: 8 independent parameters
matrix required to be non-singular

For projective transforms in
P^1 3 independent parameters
P^3 15 independent parameters